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Matrix Theory Compactification on Noncommutative $\mathbb{T}^4/\mathbb{Z}_2$

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ABSTRACT

In this paper, we construct gauge bundles on a noncommutative toroidal orbifold $\mathbb{T}^4_\theta/\mathbb{Z}_2$. First, we explicitly construct a bundle with constant curvature connections on a noncommutative $\mathbb{T}^4_\theta$ following Rieffel's method. Then, applying the appropriate quotient conditions for its $\mathbb{Z}_2$ orbifold, we find a Connes-Douglas-Schwarz type solution of matrix theory compactified on $\mathbb{T}^4_\theta/\mathbb{Z}_2$. When we consider two copies of a bundle on $\mathbb{T}^4_\theta$ invariant under the $\mathbb{Z}_2$ action, the resulting Higgs branch moduli space of equivariant constant curvature connections becomes an ordinary toroidal orbifold $\mathbb{T}^4/\mathbb{Z}_2$.

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I. Introduction

The pioneering work of Connes, Douglas, and Schwarz (CDS) [1] revealing the equivalence between noncommutative Yang-Mills theory living on the noncommutative torus and toroidally compactified IKKT(and also BFSS) M(atrix) theory [2, 3] with the constant 3-form background field has spurred various works [4] on noncommutative geometry and M/string theory since then. It has soon been known that the T-duality of M(atrix) theory can be understood in terms of Morita equivalence of the vector bundles over noncommutative tori [5, 6].

Many of these works have been related to the torus compactification and not much has been addressed to the noncommutative orbifold case. Recently, Konechny and Schwarz [7] worked out the compactification of M(atrix) theory on the $\mathbb{Z}_2$ orbifold of the noncommutative two torus. However, physically more relevant compactification on the $\mathbb{Z}_2$ orbifold of noncommutative 4-torus, a singular K_3 surface, has not been worked out so far. In the commutative case, systems of D0-branes on the commutative orbifold $\mathbb{T}^4/\mathbb{Z}_2$ were studied in [8], [9] and it is our main objective to extend the result of [8] to the noncommutative case.

We consider the compactification in the context of IKKT M(atrix) model [2] on the orbifold $\mathbb{T}^4/\mathbb{Z}_2$ where $\mathbb{Z}_2$ acts as a central symmetry $x \mapsto -x$. Thus, we need to find a Hilbert space $\mathcal{H}$ and unitary representations of $\mathbb{Z}^4$ and $\mathbb{Z}_2$ on $\mathcal{H}$ and Hermitian operators X such that

$$U_i X_j U_i^{-1} = X_j + 2\pi \delta_i^j R_i \tag{1}$$

$$U_i X_\nu U_i^{-1} = X_\nu \tag{2}$$

$$\Omega X_i \Omega = -X_i \tag{3}$$

$$\Omega X_\nu \Omega = X_\nu, \quad \nu = 0, 5, \cdots, 9, \tag{4}$$

Following the description of [10] and [7] we can find operator relations compatible with the quotient conditions (1)-(4):

$$U_i U_j = e^{2\pi i \theta_{ij}} U_j U_i, \tag{5}$$

$$\Omega U_i \Omega = U_i^{-1}, \quad \Omega^2 = 1. \tag{6}$$

When $\theta = 0$, the relations (5), (6) describe a $\mathbb{Z}_2$ equivariant vector bundle on the $\mathbb{Z}_2$ space $\mathbb{T}^4$ and X_i specify an equivariant connection on the bundle. Now the equivariant version

of the Serre-Swan theorem indicates that there is a one-to-one correspondence between $\mathbb{Z}_2$ equivariant vector bundles on the $\mathbb{Z}_2$ space $\mathbb{T}^4$ and finitely generated projective modules over the crossed product C^*-algebra $C(\mathbb{T}^4) \rtimes_\alpha \mathbb{Z}_2$. As a noncommutative analogue we see that the relations (5), (6) imply that the Hilbert space $\mathcal{H}$ is simply a module over the crossed product algebra $C(\mathbb{T}^4_\theta) \rtimes_\alpha \mathbb{Z}_2$ or $\mathcal{A}_\theta \rtimes_\alpha \mathbb{Z}_2$, where α denotes the action of $\mathbb{Z}_2$ on $\mathcal{A}_\theta$ by involution. The crossed product $\mathcal{A}_\theta \rtimes_\alpha \mathbb{Z}_2$ is the C^*-completion of the linear space of $\mathcal{A}_\theta$-valued functions on $\mathbb{Z}_2$. Thus a general element of $\mathcal{A}_\theta \rtimes_\alpha \mathbb{Z}_2$ is a formal linear combinations of elements of the form $\prod_i U_i^{n_i} \Omega^{\epsilon_i}$, where $\epsilon_i \in \{0, 1\}$. As noted in [7], a $\mathcal{A}_\theta$-module is a finitely generated projective module if and only if its corresponding module over $\mathcal{A}_\theta \rtimes_\alpha \mathbb{Z}_2$ is finitely generated projective. Thus, bundles on a NC torus $\mathbb{T}^4_\theta$ is closely related with bundles on the noncommutative torodial orbifold $\mathbb{T}^4_\theta/\mathbb{Z}_2$.

In this paper, we find a projective module solution to the quotient conditions (1)-(4). First we calculate a CDS type solution of M(atrix) theory compactified on the noncommutative 4-torus. There, we also show explicitly that the dual tori are actually related to each other through SO(4,4|$\mathbb{Z}$) transformations. From this solution we discuss that the moduli space of constant curvature connections can be identified with ordinary 4-torus. Based on such explicit CDS type solution on noncommutative $\mathbb{T}^4$, we find its $\mathbb{Z}_2$ orbifold solutions extending the result of [8] to the noncommutative torodial orbifold $\mathbb{T}^4_\theta/\mathbb{Z}_2$.

In Section II, we review the projective modules over noncommutative torus. In Section III, we construct a projective module on noncommutative 4-torus *a la* Rieffel [11] explicitly, and find a CDS type solution of M(atrix) theory compactified on the noncommutative 4-torus. It is also shown that the dual torus is actually related via SO(4,4|$\mathbb{Z}$) transformation. In Section IV, we find a solution for the noncommutative toroidal orbifold. From this solution we study the moduli space of equivariant constant curvature connections. We conclude in Section V.

II. Noncommutative vector bundles over noncommutative torus

In this section we review noncommutative vector bundles over NC d-torus $\mathbb{T}_\theta^d$, following the lines of [12, 11, 5, 6]. Recall that $\mathbb{T}_\theta^d$ is the deformed algebra of the algebra of smooth functions on the torus $\mathbb{T}^d$ with the deformation parameter θ, which is a real $d \times d$ anti-symmetric matrix. This algebra is generated by operators $U_1, \cdots, U_d$ obeying the following relations

$$U_i U_j = e^{2\pi i \theta_{ij}} U_j U_i \quad \text{and} \quad U_i^* U_i = U_i U_i^* = 1, \quad i, j = 1, \cdots, d.$$

The above relations define the presentation of the involutive algebra

$$\mathcal{A}_\theta^d = \{ \sum a_{i_1 \cdots i_d} U_1^{i_1} \cdots U_d^{i_d} \mid a = (a_{i_1 \cdots i_d}) \in \mathcal{S}(\mathbb{Z}^d) \}$$

where $\mathcal{S}(\mathbb{Z}^d)$ is the Schwartz space of sequences with rapid decay. According to the dictionary in [13], the construction of a noncommutative vector bundle over $\mathbb{T}_\theta^d$ corresponds to the construction of finitely generated projective modules over $\mathcal{A}_\theta^d$. It was proved in [11] that every projective module over a smooth algebra $\mathcal{A}_\theta^d$ can be represented by a direct sum of modules of the form $\mathcal{S}(\mathbb{R}^p \times \mathbb{Z}^q \times F)$, the linear space of Schwartz functions on $\mathbb{R}^p \times \mathbb{Z}^q \times F$, where $2p+q = d$ and F is a finite abelian group. The module action is specified by operators on $\mathcal{S}(\mathbb{R}^p \times \mathbb{Z}^q \times F)$ and the commutation relation of these operators should be matched with that of elements in $\mathcal{A}_\theta^d$.

On such bundles or modules there are notions of connections and the Chern character [1, 5, 6, 14]. Recall that there is the dual action of the torus group $\mathbb{T}^d$ on $\mathcal{A}_\theta^d$ which gives a Lie group homomorphism of $\mathbb{T}^d$ into the group of automorphisms of $\mathcal{A}_\theta^d$. Its infinitesimal form generates a homomorphism of Lie algebra L of $\mathbb{T}^d$ into Lie algebra of derivations of $\mathcal{A}_\theta^d$. Note that the Lie algebra L is abelian and is isomorphic to $\mathbb{R}^d$. Let $\delta : L \to \text{Der}\ (\mathcal{A}_\theta^d)$ be the homomorphism. For each $X \in L$, $\delta(X) := \delta_X$ is a derivation i.e., for $u, v \in \mathcal{A}_\theta^d$,

$$\delta_X(uv) = \delta_X(u)v + u\delta_X(v).$$

Derivations corresponding to the generators $\{e_1, \cdots, e_d\}$ of L will be denoted by $\delta_1, \cdots, \delta_d$.

For the generators U_i's of $\mathbb{T}^d_\theta$, it has the following property

$$\delta_i(U_j) = 2\pi i \delta_{ij} \cdot U_j.$$

If E is a projective $\mathcal{A}^d_\theta$-module, a connection ∇ on E is a linear map from E to $E \otimes L^*$ such that for all $X \in L$,

$$\nabla_X(\xi u) = (\nabla_X \xi)u + \xi \delta_X(u), \quad \xi \in E, u \in \mathcal{A}^d_\theta.$$

It is easy to see that

$$[\nabla_i, U_j] = 2\pi i \delta_{ij} \cdot U_j.$$

Furthermore, for an $\mathcal{A}^d_\theta$-valued inner product $\langle \cdot, \cdot \rangle$ on E, if ∇ has the property that

$$\langle \nabla_X \xi, \eta \rangle + \langle \xi, \nabla_X \eta \rangle = \delta_X(\langle \xi, \eta \rangle),$$

then it is called a Hermitian connection. The curvature $\mathcal{F}_\nabla$ of a connection ∇ is a 2-form on L with values in the algebra of endomorphisms of E. That is, for $X, Y \in L$,

$$\mathcal{F}_\nabla(X, Y) := [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.$$

Since L is abelian, we simply have $\mathcal{F}_\nabla(X, Y) = [\nabla_X, \nabla_Y]$. Denote by $\mathcal{E} = \mathrm{End}_{\mathcal{A}_\theta}(E)$ the algebra of endomorphisms of E. Note that if ∇ and ∇' are two Hermitian connections, then $\nabla_X - \nabla'_X$ belongs to the algebra $\mathcal{E}$. Thus once we have fixed a connection ∇, then every other connections is of the form $\nabla + A$, here A is a linear map L into $\mathcal{E}$. In other words, the space of Hermitian connections is an affine space with vector space consisting of the linear maps from L to $\mathcal{E}$ and also the algebra is related with a moduli space of a certain connections.

We now consider the endomorphisms algebra of a module over $\mathcal{A}^d_\theta$. Let Λ be a lattice in $H = M \times \widehat{M}$, where $M = \mathbb{R}^p \times \mathbb{Z}^q \times F$ and $\widehat{M}$ is its dual. Let T be the corresponding embedding map in the sense of [11]. Thus Λ is the image of $\mathbb{Z}^d$ under the map T and this determines a projective module which will be denoted by E_Λ. Consider the lattice

$$\Lambda^\perp := \{(m, \hat{s}) \in M \times \widehat{M} \mid \theta((m, \hat{s}), (n, \hat{t})) = \hat{t}(m) - \hat{s}(n) \in \mathbb{Z}, \quad \text{for all } (n, \hat{t}) \in \Lambda\}.$$

From the definition, it is easy to see that every operator of the form

$$\mathcal{U}_{(m,\hat{s})} = (n) = e^{2\pi i \hat{s}(n)} f(n+m)$$

for $(m, \hat{s}) \in \Lambda^{\perp}$, commutes with all operators $\mathcal{U}_{(n,\hat{t})}$, $(n, \hat{t}) \in \Lambda$. In fact one can show that the algebra of endomorphisms on E_Λ, denoted by $\mathrm{End}_{\mathcal{A}_\theta}(E_\Lambda)$, is a C^*-algebra which is obtained by C^*-completion of the space spanned by operators $\mathcal{U}_{(m,\hat{s})}$, $(m, \hat{s}) \in \Lambda^{\perp}$. As shown in [11], the algebra $\mathrm{End}_{\mathcal{A}_\theta}(E_\Lambda)$ can be identified with a noncommutative torus $\mathcal{A}_{\widehat{\theta}}$, here $\widehat{\theta}$ is a bilinear form on $\Lambda^{\perp}$,i.e., $\mathcal{A}_{\widehat{\theta}}$ is Morita equivalent to $\mathcal{A}_\theta$. Recall that a C^*-algebra A is said to be (strongly) Morita equivalent to A' if $A' \cong \mathrm{End}_A(E)$ for some finite projective module E. In general, as was proved in [6], a NC torus $\mathcal{A}_{\tilde{\theta}}$ is Morita equivalent to $\mathcal{A}_\theta$ if θ and $\tilde{\theta}$ are related by $\tilde{\theta} = (A\theta + B)(C\theta + D)^{-1}$, where $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \ \mathrm{SO}(d, d|\mathbb{Z})$.

We shall now turn to the description of the Chern character. In general $K_0(\mathcal{A}_\theta^d)$ classifies projective modules over $\mathcal{A}_\theta^d$. In fact the positive cone $K_0^+(\mathcal{A}_\theta^d)$ corresponds to genuine projective modules and if θ is not rational, $K_0^+(\mathcal{A}_\theta^d)$ consists exactly of its elements of strictly positive trace. The Chern character of a gauge bundle on a noncommutative torus is an element in the Grassmann algebra $\wedge^{\cdot}(L^*)$, where L denotes the Lie algebra of $\mathbb{T}^d$ and L^* is the dual vector space of L. Since there is a lattice D in L, we see that there are elements of $\wedge^{\cdot} D^*$ which are integral. Now the Chern character is the map $\mathrm{Ch} : K_0(\mathcal{A}_\theta^d) \to \wedge^{\mathrm{ev}}(L^*)$ defined by

$$\mathrm{Ch}(E) := \widehat{\tau}(e^{\frac{\mathcal{F}}{2\pi i}}) = \sum_{k=0} \frac{1}{(2\pi i)^k} \frac{\widehat{\tau}(\mathcal{F}^k)}{k!},$$

where E is any gauge bundle and $\mathcal{F}$ is a curvature of an arbitrary connection on E and $\widehat{\tau}$ is a trace on the algebra of endomorphisms. In general the Chern character is integral in the commutative case. This is no longer true for the noncommutative case. However, in the case of noncommutative torus, there is an integral element related to the Chern character by the formula

$$\mathrm{Ch}(E) = e^{i(\theta)} \mu(E). \tag{7}$$

Here $i(\theta)$ denotes the contraction with the deform parameter θ regarded as an element of $\wedge^2 L$. The formula (7) can be realized as a noncommutative generalization of Mukai vector.

In particular, $\mu(E) = e^{-i(\theta)}\mathrm{Ch}(E)$ is an integral element of $\wedge^{\cdot}(L^*)$ which is related with the Chern character on the classical torus. Also once we fix the deformation parameter, then the Chern character $\mathrm{Ch}(E)$ is completely determined by its integral part $\mu(E)$. Note that if the 0th component of the Chern character or the trace is strictly positive, then the gauge bundle E belongs to the positive cone of $K_0(\mathcal{A}_\theta^d)$ and hence it can be written as a direct sum of the form $\mathcal{S}(\mathbb{R}^p \times \mathbb{Z}^q \times F)$, [11].

III. Compactification on noncommutative $\mathbb{T}^4$.

In this section we study the compactification solutions on a noncommutative 4-torus $\mathbb{T}_\theta^4$ for the case $e^{2\pi i\theta_{ij}} \neq 1$, following the guide line in [1]. After we fix U_1, U_2, U_3 and U_4, or a projective module, the general solution has the form of $X_i = \bar{X}_i + A_i$, where $\bar{X}_i$ are particular solutions and A_i are operators commuting with U_i. Here we consider a projective module of the form $\mathcal{S}(\mathbb{R}^p \times \mathbb{Z}^q) \otimes \mathcal{S}(F)$, where $2p + q = 4$. Thus there are three types of modules over $\mathcal{A}_\theta$ according to $p = 0, 1, 2$. When $p = 0$, it is a free module. The other two types are of the form $\mathcal{S}(\mathbb{R} \times \mathbb{Z}^2) \otimes \mathcal{S}(F)$ and $\mathcal{S}(\mathbb{R}^2) \otimes \mathcal{S}(F)$. As is discussed in Section II, a gauge bundle on $\mathbb{T}_\theta^4$ correspond to an element of positive trace which is the 0th component of the Chern character and the Chern character is determined by its integral part μ. Thus it is natural to start with the construction on $\mathcal{S}(F)$ to describe projective modules. Here we will only consider the case when $p = 2$ which is related with (4220)-systems with a constant curvature considered in [15, 16]. Let $F = \mathbb{Z}_{M_1} \times \mathbb{Z}_{M_2}$, where $\mathbb{Z}_{M_i} = \mathbb{Z}/M_i\mathbb{Z}$, $(i = 1, 2)$ and consider the space $\mathbb{C}^{M_1} \otimes \mathbb{C}^{M_2}$ as the space of functions on $C(\mathbb{Z}_{M_1} \times \mathbb{Z}_{M_2})$. For all $M_i \in \mathbb{Z}$ and $N_i \in \mathbb{Z}/M_i\mathbb{Z}$ such that M_i and N_i are relatively prime, define operators W_i on $C(\mathbb{Z}_{M_1} \times \mathbb{Z}_{M_2})$ by

$$
\begin{aligned}
(W_1 f)(k_1, k_2) &= f(k_1 - N_1, k_2) \\
(W_2 f)(k_1, k_2) &= \exp(-\frac{2\pi i k_1}{M_1}) f(k_1, k_2) \\
(W_3 f)(k_1, k_2) &= f(k_1, k_2 - N_2) \\
(W_4 f)(k_1, k_2) &= \exp(-\frac{2\pi i k_2}{M_2}) f(k_1, k_2).
\end{aligned}
$$

The operators satisfy the commutation relation

$$W_1 W_2 = \exp(2\pi i \frac{N_1}{M_1}) W_2 W_1$$
$$W_3 W_4 = \exp(2\pi i \frac{N_2}{M_2}) W_4 W_3,$$

otherwise commuting. If we write $W_i W_j = \exp(2\pi i \psi_{ij}) W_j W_i$, then the antisymmetric 4×4 matrix $\psi = (\psi_{ij})$ is of the form

$$\psi = \begin{pmatrix} 0 & \frac{N_1}{M_1} & 0 & 0 \\ -\frac{N_1}{M_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{N_2}{M_2} \\ 0 & 0 & -\frac{N_2}{M_2} & 0 \end{pmatrix}. \tag{8}$$

Let $T : \mathbb{Z}^4 \longrightarrow \mathbb{R}^2 \times \mathbb{R}^{2*}$ be an embedding map. Thus its matrix representation $T = \Big(x_{ij}\Big)$, $i, j = 1, \cdots, 4$, has nonzero determinant and satisfies $(\wedge^2 T^*)(\omega) = -\gamma$ where $\omega = e_3 \wedge e_1 + e_4 \wedge e_2 \in \wedge^2(\mathbb{Z}^4)$ and e_i are standard basis for $\mathbb{Z}^4$. Equivalently, if we consider the Heisenberg representation of $\mathbb{Z}^4$ in a Hilbert space, the desired operators acting on the space of smooth functions on $\mathbb{R}^2$ are defined by the following form:

$$(V_i f)(s_1, s_2) = (V_{e_i} f)(s_1, s_2) := \exp(2\pi i (s_1 x_{3i} + s_2 x_{4i})) f(s_1 + x_{1i}, s_2 + x_{2i}).$$

These operators obey the commutation relation

$$V_i V_j = e^{-2\pi i \gamma_{ij}} V_j V_i,$$

where

$$\gamma_{ij} = \begin{vmatrix} x_{1i} & x_{1j} \\ x_{3i} & x_{3j} \end{vmatrix} + \begin{vmatrix} x_{2i} & x_{2j} \\ x_{4i} & x_{4j} \end{vmatrix}.$$

Since γ is a real matrix, the operators V_i act on the Schwartz space $\mathcal{S}(\mathbb{R}^2)$. Now we define operators $U_i = V_i \otimes W_i$ acting on the space $E_T := \mathcal{S}(\mathbb{R}^2) \otimes \mathbb{C}^{M_1} \otimes \mathbb{C}^{M_2}$ as follows

$$(U_1 f)(s_1, s_2, k_1, k_2) = e^{2\pi i (s_1 x_{31} + s_2 x_{41})} f(s_1 + x_{11}, s_2 + x_{21}, k_1 - N_1, k_2)$$
$$(U_2 f)(s_1, s_2, k_1, k_2) = e^{2\pi i (s_1 x_{32} + s_2 x_{42})} \cdot e^{-\frac{2\pi i k_1}{M_1}} f(s_1 + x_{12}, s_2 + x_{22}, k_1, k_2)$$
$$(U_3 f)(s_1, s_2, k_1, k_2) = e^{2\pi i (s_1 x_{33} + s_2 x_{43})} f(s_1 + x_{13}, s_2 + x_{23}, k_1, k_2 - N_2)$$
$$(U_4 f)(s_1, s_2, k_1, k_2) = e^{2\pi i (s_1 x_{34} + s_2 x_{44})} \cdot e^{-\frac{2\pi i k_2}{M_2}} f(s_1 + x_{14}, s_2 + x_{24}, k_1, k_2).$$

Then it is easy to see that they satisfy

$$U_iU_j = \exp(-2\pi i\gamma_{ij} + 2\pi i\psi_{ij})U_jU_i.$$

Thus we have solution of (5) if $\gamma = \psi - \theta$.

Consider operators $\bar{X}_i$ acting on $E_T = \mathcal{S}(\mathbb{R}^2) \otimes \mathbb{C}^{M_1} \otimes \mathbb{C}^{M_2}$ given by

$$\begin{aligned}(\bar{X}_i f)(s_1, s_2, k_1, k_2) &= 2\pi i A_i^1 s_1 f(s_1, s_2, k_1, k_2) + 2\pi i A_i^2 s_2 f(s_1, s_2, k_1, k_2) \\ &\quad - A_i^3 \frac{\partial f(s_1, s_2, k_1, k_2)}{\partial s_1} - A_i^4 \frac{\partial f(s_1, s_2, k_1, k_2)}{\partial s_2},\end{aligned} \tag{9}$$

where A_i^k are any real numbers yet to be determined. From the definition of U_i and $\bar{X}_i$, it is easy to see that the operators W_i are commute with $\bar{X}_i$. Suppose that the operators $\bar{X}_i$ satisfy the equation (1), i.e.,

$$U_i\bar{X}_jU_i^{-1} = \bar{X}_j + 2\pi\delta_i^j R_i.$$

By a straightforward calculation, the constant matrix (A_i^j) in (9) can be obtained as in the following form:

$$\left(R_iA_i^j\right) T = -i \text{ Id}.$$

Since the inverse matrix of T can be written as

$$T^{-1} = \frac{1}{\det T}\left((-1)^{i+j}B_{ji}\right),$$

where B_{ij} is the (ij)-minor of the matrix T, we see that

$$A_i^k = (-1)^{i+k} \cdot \frac{R_i}{i} \cdot \frac{1}{\det T} \cdot B_{ki}, \tag{10}$$

and this gives a particular solution to the equations (2) and (3). It is easy to check that the commutator has of the form

$$[\bar{X}_i, \bar{X}_j] = 2\pi i \left(\begin{vmatrix} A_i^1 & A_i^3 \\ A_j^1 & A_j^3 \end{vmatrix} + \begin{vmatrix} A_i^2 & A_i^4 \\ A_j^2 & A_j^4 \end{vmatrix} \right).$$

By (10), we have

$$\begin{aligned}[\bar{X}_i, \bar{X}_j] &= -2\pi i \cdot \frac{R_iR_j}{(\det T)^2} \left\{ (-1)^{i+1}(-1)^{j+1} \begin{vmatrix} B_{1i} & B_{3i} \\ B_{1j} & B_{3j} \end{vmatrix} + (-1)^i(-1)^j \begin{vmatrix} B_{2i} & B_{4i} \\ B_{2j} & B_{4j} \end{vmatrix} \right\} \\ &= 2\pi i(-1)^{i+j+1} \cdot \frac{R_iR_j}{(\det T)^2} \left\{ \begin{vmatrix} B_{1i} & B_{3i} \\ B_{1j} & B_{3j} \end{vmatrix} + \begin{vmatrix} B_{2i} & B_{4i} \\ B_{2j} & B_{4j} \end{vmatrix} \right\} \\ &= 2\pi i(-1)^{i+j+1} \cdot \frac{R_iR_j}{\det T} \cdot *\gamma_{ij}.\end{aligned}$$

Now we should find generators of the set of operators which commute with U_i's. To find such operators we need to describe an embedding map which corresponds to the dual lattice of the lattice defined by the embedding map T as discussed in Section II. For such a map, let

$$S = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \cdot (T^t)^{-1} = \frac{1}{\det T} \begin{pmatrix} B_{31} & -B_{32} & B_{33} & -B_{34} \\ -B_{41} & B_{42} & -B_{43} & B_{44} \\ -B_{11} & B_{12} & -B_{13} & B_{14} \\ B_{21} & -B_{22} & B_{23} & -B_{24} \end{pmatrix}. \tag{11}$$

Using the matrix (11), we define operators acting on E_T by

$$\begin{aligned}
(Z_1 f)(s_1, s_2, k_1, k_2) &= e^{\frac{2\pi i(-s_1 B_{11} + s_2 B_{21})}{M_1|T|}} \cdot e^{\frac{2\pi i b_1 k_1}{M_1}} f(s_1 + \frac{B_{31}}{M_1|T|}, s_2 - \frac{B_{41}}{M_1|T|}, k_1, k_2) \\
(Z_2 f)(s_1, s_2, k_1, k_2) &= e^{\frac{2\pi i(s_1 B_{12} - s_2 B_{22})}{M_1|T|}} f(s_1 - \frac{B_{32}}{M_1|T|}, s_2 + \frac{B_{42}}{M_1|T|}, k_1 - 1, k_2) \\
(Z_3 f)(s_1, s_2, k_1, k_2) &= e^{\frac{2\pi i(-s_1 B_{13} + s_2 B_{23})}{M_2|T|}} \cdot e^{\frac{2\pi i b_2 k_1}{M_2}} f(s_1 + \frac{B_{33}}{M_2|T|}, s_2 - \frac{B_{43}}{M_2|T|}, k_1, k_2) \\
(Z_4 f)(s_1, s_2, k_1, k_2) &= e^{\frac{2\pi i(s_1 B_{14} - s_2 B_{24})}{M_2|T|}} f(s_1 - \frac{B_{34}}{M_2|T|}, s_2 + \frac{B_{44}}{M_2|T|}, k_1, k_2 - 1),
\end{aligned}$$

where $|T| = \operatorname{Pf}(\psi - \theta)$ denotes the determinant of T and b_1, b_2 are integers such that $a_i M_i + b_i N_i = 1$, a_i are also integers. To check the operators Z_i commute with all U_j's, let $Z_i U_j = e^{2\pi i \lambda_{ij}} U_j Z_i$. Then it is easy to see that

$$\lambda_{ij} = \frac{1}{M_k|T|} \left\{ \begin{vmatrix} x_{1i} & x_{3i} \\ (-1)^{3+j} B_{3j} & (-1)^{1+j} B_{1j} \end{vmatrix} + \begin{vmatrix} x_{2i} & x_{4i} \\ (-1)^{4+j} B_{4j} & (-1)^{2+j} B_{2j} \end{vmatrix} \right\} - \delta_{ij} \frac{b_k N_k}{M_k}, \tag{12}$$

where $k = 1, 2$ depending on ij. From the relation (12),

$$\begin{aligned}
\lambda_{ij} &= 0 \quad \text{when } i \neq j \\
\lambda_{ii} &= \frac{1}{M_k} - \frac{b_k N_k}{M_k} = \frac{-a_k M_k}{M_k} = -a_k \in \mathbb{Z}.
\end{aligned}$$

Thus Z_i commute with all U_j's.

Furthermore the operators satisfy

$$Z_i Z_j = e^{2\pi i \hat{\theta}} Z_j Z_i. \tag{13}$$

Now $\hat{\theta}$ can be calculated directly and it is given by

$$\begin{aligned}
\hat{\theta}_{12} &= \frac{a_1 N_2 + b_1 N_2 \theta_{12} + a_1 M_2 \theta_{34} - b_1 M_2 \mathrm{Pf}(\theta)}{M_1 M_2 \mathrm{Pf}(\psi - \theta)} \\
\hat{\theta}_{13} &= \frac{\theta_{13}}{M_1 M_2 \mathrm{Pf}(\psi - \theta)} \\
\hat{\theta}_{14} &= \frac{\theta_{14}}{M_1 M_2 \mathrm{Pf}(\psi - \theta)} \\
\hat{\theta}_{23} &= \frac{\theta_{23}}{M_1 M_2 \mathrm{Pf}(\psi - \theta)} \\
\hat{\theta}_{24} &= \frac{\theta_{24}}{M_1 M_2 \mathrm{Pf}(\psi - \theta)} \\
\hat{\theta}_{34} &= \frac{a_2 N_1 + b_2 N_1 \theta_{34} + a_2 M_1 \theta_{12} - b_2 M_1 \mathrm{Pf}(\theta)}{M_1 M_2 \mathrm{Pf}(\psi - \theta)}.
\end{aligned}$$

Also we have

$$\hat{\theta} = (A\theta + B)(N - M\theta)^{-1} \tag{14}$$

where

$$A = \begin{pmatrix} 0 & -a_1 & 0 & 0 \\ a_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_2 \\ 0 & 0 & a_2 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} b_1 & 0 & 0 & 0 \\ 0 & b_1 & 0 & 0 \\ 0 & 0 & b_2 & 0 \\ 0 & 0 & 0 & b_2 \end{pmatrix}$$

and

$$N = \begin{pmatrix} N_1 & 0 & 0 & 0 \\ 0 & N_1 & 0 & 0 \\ 0 & 0 & N_2 & 0 \\ 0 & 0 & 0 & N_2 \end{pmatrix} \quad M = \begin{pmatrix} 0 & M_1 & 0 & 0 \\ -M_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & M_2 \\ 0 & 0 & -M_2 & 0 \end{pmatrix}.$$

From the equation (14), we see that $-\theta$ and $\hat{\theta}$ are related by $\mathrm{SO}(4,4|\mathbb{Z})$ transformation.

Note that $\mathrm{U}(n)$ theory on $\mathcal{A}_{-\theta}$ is equivalent to $\mathrm{U}(1)$ theory on $\mathcal{A}_{\hat{\theta}}$. For $\mathrm{U}(1)$ theory the generators Z_i can be identified with functions on the dual torus:

$$Z_j \to e^{i\sigma_j}$$

where σ_j are coordinates of the dual torus such that

$$[\sigma_i, \sigma_j] = -2\pi i \hat{\theta}_{ij}.$$

Now the general solution of the compactification is given by

$$X_i = \bar{X}_i + \sum_{i_1,\cdots,i_4\in\mathbb{Z}} \Psi_{i_1i_2i_3i_4} Z_1^{i_1} Z_2^{i_2} Z_3^{i_3} Z_4^{i_4},$$

where the coefficients $\Psi_{i_1i_2i_3i_4}$ are c-numbers.

Recall that a connection in a module E_T is determined by a set of operators $\nabla_1, \cdots, \nabla_4$ in E_T such that

$$[\nabla_i, U_j] = 2\pi i \delta_{ij} U_j.$$

From the definition of $\bar{X}_i$ given in (9) we have

$$[\bar{X}_i, U_j] = -2\pi\delta_{ij} R_j U_j.$$

Thus we see that the special solution $\bar{X}_i$ is related with connections by $\bar{X}_i = \frac{R_i}{i}\nabla_i$ and for such connection ∇, the constant curvature $\mathcal{F} = (\mathcal{F}_{ij})$ is given by

$$\mathcal{F} = \gamma^{-1} \cdot \mathrm{Id}_N, \qquad \text{where } N = N_1N_2. \tag{15}$$

Now the general solution should be identified as

$$X_i = \frac{R_i}{i}\nabla_i + A_i(\sigma_1, \sigma_2, \sigma_3, \sigma_4) \tag{16}$$

where A_i are gauge fields defined on a noncommutative torus.

Note that from the curvature form (15), it corresponds to the $U(N)$ gauge theory with vanishing $su(N)$ curvature. This type of solutions has been studied in [17] for noncommutative $\mathbb{T}^2$ and in [18] for higher torus case. This was generalized to a nonvanishing $su(N)$ curvature case in [15] and it has been noted that the analysis for noncommutative tori is the same as that of [16] for commutative tori. In fact the above solution has been described by (4220) system with trivial $SU(N)$ gauge fields in [16] and its moduli space can be identified with $\mathbb{T}^4$. So we may expect that the moduli space of constant curvature connections in noncommutative torus is of the same form as in the ordinary torus.

The operators

$$\widetilde{\nabla}_j = \frac{i}{R_j}\bar{X}_j + \alpha_j, \qquad j = 1, \cdots, 4, \tag{17}$$

where α_j is any real number, determine a Hermitian connection with constant curvature in E_T. Furthermore connections of the form (16) define a representation on $L^2(\mathbb{R}^2, \mathbb{C}^{M_1} \otimes \mathbb{C}^{M_2})$ of the Heisenberg commutation relations and from this one can follow the same steps in [12] to show that connections of the form (17) can be found in each gauge orbits and two such connections $\frac{i}{R_j}\bar{X}_j + \alpha_j$ and $\frac{i}{R_j}\bar{X}_j + \mu_j$ are gauge equivalent if and only if $\alpha_j - \mu_j \in \mathbb{Z}$. Thus the moduli space of constant curvature connections can be identified with $(\mathbb{R}/\mathbb{Z})^4 \cong (S^1)^4 \cong \mathbb{T}^4$. In general, if we consider a projective module consisting of n copies of such modules, such as $E_{T_1} \oplus \cdots \oplus E_{T_n}$, where T_i is an embedding, then there is a constant curvature connection on each summand such that the overall curvature is given by $\mathcal{F} = \oplus \mathcal{F}_k$, where $\mathcal{F}_k$ is given as in (15) with the same γ. Thus for a constant curvature connection on E which breaks a projective module E into $\oplus_k E_{T_i}$, block diagonal construction gives the moduli space of the form $(\mathbb{T}^4)^n / S_n$, where S_n is the symmetric group.

IV. Compactification on noncommutative toroidal orbifold $\mathbb{T}_\theta/\mathbb{Z}_2$

In this section we find solutions for the quotient conditions (1)-(4) along with the projective module actions (5) and (6) via the compactification solutions on a noncommutative torus $\mathbb{T}^4_\theta$ obtained in Section III. From this we find the moduli space of equivariant constant curvature connections on noncommutative toroidal orbifold $\mathbb{T}^4_\theta/\mathbb{Z}_2$.

Consider the module $E_T := \mathcal{S}(\mathbb{R}^2) \otimes C(\mathbb{Z}_{M_1}) \otimes C(\mathbb{Z}_{M_2})$ together with U_i's as operators acting on it. The general solution for the quotient conditions has been identified as

$$X_j = \frac{R_j}{i}\nabla_j + A_j(\sigma_1, \sigma_2, \sigma_3, \sigma_4), \qquad 1 \leq j \leq 4. \tag{18}$$

To find solutions for the quotient conditions on the compactified part we need to solve for Ω which satisfies $\Omega U_i \Omega = U_i^{-1}$ and $\Omega^2 = 1$. Consider an operator Ω_0 on E_T defined by

$$(\Omega_0 f)(s_1, s_2, k_1, k_2) = f(-s_1, -s_2, -k_1, -k_2).$$

It is easy to see that $\Omega_0 U_i \Omega_0 U_i = e^{2\pi i(x_{1i}x_{3i} + x_{2i}x_{4i})}$. By redefining $U_i \mapsto e^{-\pi i(x_{1i}x_{3i} + x_{2i}x_{4i})} U_i$, we get $\Omega_0 U_i \Omega_0 = U_i^{-1}$ and $\Omega_0^2 = 1$. Thus we have a solution for (6) i.e., Ω_0 together with U_i's

define a projective module over $\mathcal{A}_\theta \rtimes \mathbb{Z}_2$. As was indicated in [7], there might be other $\mathbb{Z}_2$ actions on the module. To get other actions on the module, consider the operators Z_i defined in Section III. As for the U_i's, rescale Z_i by $e^{-\pi i(B_{1i}B_{3i}+B_{2i}B_{4i})}Z_i$ and we get the relation

$$\Omega_0 Z_i \Omega_0 = Z_i^{-1}. \tag{19}$$

Since Z_i commute with all U_j's, the operators $\Omega_{n_1\cdots n_4} = e^{i\phi}\Omega_0 Z_1^{n_1} Z_2^{n_2} Z_3^{n_3} Z_4^{n_4}$, $(n_i \in \mathbb{Z})$, satisfy the equation (6), where ϕ is a phase which is chosen to get the relation $\Omega^2 = 1$ and it can be calculated explicitly by using the commutation relations given in (13). Now consider the general solution (18) satisfying (1) and (2). Recall $\nabla_i = \frac{i}{R_i}\bar{X}_i$. For $\bar{X}_i$, which was defined in (9), it is easy to verify that $\Omega_0 \bar{X}_i \Omega_0 = -\bar{X}_i$. But since $\bar{X}_i$ do not commute with Z_i's, we see that Ω_0 is the unique solution for the equation $\Omega \bar{X}_i \Omega = -\bar{X}_i$. By definition of the functions A_i on the dual torus and by the relation (19), we have $\Omega_0 A_i(\sigma_1, \sigma_2, \sigma_3, \sigma_4)\Omega_0 = A_i(-\sigma_1, -\sigma_2, -\sigma_3, -\sigma_4)$. Applying Ω_0 to the both sides on the equation (18) we see that

$$A_i(-\sigma_1, -\sigma_2, -\sigma_3, -\sigma_4) = -A_i(\sigma_1, \sigma_2, \sigma_3, \sigma_4), \tag{20}$$

which implies that the functions A_i are odd functions. If we consider a constant curvature connection ∇ on E_T, the functions A_i in (20) can be represented by a real constant and hence it vanishes. In other words the moduli space has no Higgs branch. Note that this type of solutions has been studied in [8] for the ordinary torodial orbifold $\mathbb{T}^4/\mathbb{Z}_2$ under the name of **Rep. II**.

In the above representation, the moduli space of constant curvature connections on E_T over $\mathbb{T}^4_\theta$ is not preserved by the $\mathbb{Z}_2$ action on E_T. So it may be more natural to consider two copies of E_T which respect the $\mathbb{Z}_2$ action and this corresponds to **Rep. I** of [8]. Consider the bundle of the form $E_T^2 = E_T \oplus E_T$ and define operators acting on E_T^2 by

$$\Omega = \begin{pmatrix} \Omega_0 & 0 \\ 0 & -\Omega_0 \end{pmatrix}, \quad \text{and} \quad \mathbf{U}_i = \begin{pmatrix} U_i & 0 \\ 0 & U_i \end{pmatrix},$$

where Ω_0 and U_i's are operators on E_T given as above and in Section III. Then it is easy to check that

$$\begin{aligned} \mathbf{U}_i\mathbf{U}_j &= e^{2\pi i\theta_{ij}}\mathbf{U}_j\mathbf{U}_i, \\ \Omega\mathbf{U}_i\Omega &= \mathbf{U}_i^{-1} \quad \text{and} \quad \Omega^2 = 1. \end{aligned} \tag{21}$$

Thus the relations (21) defines a projective module over $\mathcal{A}_\theta \rtimes \mathbb{Z}_2$. Since $\bar{X}_i$ defines a particular solution, we may write the general solution on the torus as follows

$$X_i = \bar{X}_i + \begin{pmatrix} A_i^{11} & A_i^{12} \\ A_i^{21} & A_i^{22} \end{pmatrix}.$$

Since the matrix $\begin{pmatrix} A_i^{11} & A_i^{12} \\ A_i^{21} & A_i^{22} \end{pmatrix}$ should commute with all the $\mathbf{U}_i$'s, each entries A_i^{jk} commute with U_i's. In other words, the operators A_i^{jk} are generated by Z_i's. Thus they can be identified with functions on the dual torus. Now the general solutions should be identified as

$$X_i = \frac{R_i}{i}\nabla_i + \begin{pmatrix} A_i^{11}(\sigma_j) & A_i^{12}(\sigma_j) \\ A_i^{21}(\sigma_j) & A_i^{22}(\sigma_j) \end{pmatrix}. \tag{22}$$

By applying Ω we find

$$\begin{pmatrix} A_i^{11}(-\sigma_j) & A_i^{12}(-\sigma_j) \\ A_i^{21}(-\sigma_j) & A_i^{22}(-\sigma_j) \end{pmatrix} = \begin{pmatrix} -A_i^{11}(\sigma_j) & A_i^{12}(\sigma_j) \\ A_i^{21}(\sigma_j) & -A_i^{22}(\sigma_j) \end{pmatrix}.$$

Note that the diagonal entries of the matrix in (22) are odd functions on the dual torus, and this fact will be used in finding the moduli space below. Meanwhile the off-diagonal entries are even fuctions of σ. Here, the gauge transformation should be invariant under Ω implementing the $\mathbb{Z}_2$ quotient condition. This implies that the gauge parameter in general should be given by $\Lambda = \begin{pmatrix} \lambda_{ev}^{11} & \lambda_{od}^{12} \\ \lambda_{od}^{21} & \lambda_{ev}^{22} \end{pmatrix}$ where the subscript ev or od indicates an even or odd function of σ. This indicates us that not all the $U(2)$ group acts. We now consider the constant curvature connection ∇ on E_T considered in Section III. In this case, as discussed in **Rep. II** we have constant gauge field in (22). Thus the diagonal entries vanish and the bundle becomes singular at the fixed points. For the ordinary case this has been related to the existence of two-brane charge at the collapsing two-cycle of the blown-up space [19, 20, 21].

Now the solutions of the constant curvature connection in this case are given by

$$X_i = \bar{X}_i + \begin{pmatrix} 0 & A_i^{12}(\sigma_j) \\ A_i^{12\dagger}(\sigma_j) & 0 \end{pmatrix}.$$

One of the A_i components can be gauged away by constant gauge transformation of the type $\begin{pmatrix} \lambda & 0 \\ 0 & \hat{\lambda} \end{pmatrix}$ which can be decomposed into two parts, one propotional to the identity and

the other proportional to $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Since we are only considering constant gauge transformations in dealing with the moduli space, the noncommutativity does not affect the result as in Section III. The remaining component of A_i has translational symmetry of the commutative 4-torus. This fact together with a residual guage symmetry σ_3 now yields a Higgs branch moduli space of constant curvature connections to be an ordinary torodial orbifold $\mathbb{T}^4/\mathbb{Z}_2$.

For the uncompactified X_ν sector, the solution is the same as in the commutative case [8]; the moduli becomes $\mathbb{R}^5 \times \mathbb{R}^5$ when $A_i = 0$, and when $A_i \neq 0$ the transverse moduli becomes $\mathbb{R}^5$ for generic points in $\mathbb{T}^4/\mathbb{Z}_2$, and $\mathbb{R}^5 \times \mathbb{R}^5$ at the fixed points in $\mathbb{T}^4/\mathbb{Z}_2$. Thus this can be viewed as a fibration over the Higgs branch of $\mathbb{T}^4/\mathbb{Z}_2$, with the fiber $\mathbb{R}^5$ at a generic point and with the fiber $\mathbb{R}^5 \times \mathbb{R}^5$ at the orbifold fixed points as suggested in the commutative case [8].

For the ordinary $\mathbb{T}^4$, the discussion above corresponds to the construction of the theory of zero branes on $\mathbb{T}^4/\mathbb{Z}_2$. We first considered a T-duality on the covering torus $\mathbb{T}^4$ to a dual torus $\hat{\mathbb{T}}^4$ and then project to $\hat{\mathbb{T}}^4/\mathbb{Z}_2$. So, for N identical D0-branes on $\mathbb{T}^4/\mathbb{Z}_2$ we need $2N$ zero branes on $\mathbb{T}^4$. This is described by $U(2N)$ gauge theory and the gauge group is broken down to $U(N) \times U(N)$. In [8], it has been shown that the moduli space of the flat connections is identified with $\mathbb{T}^4/\mathbb{Z}_2$. In fact our above analysis on the moduli space of constant curvature connections is exactly the same as the one in [8].

V. Conclusion and prospect

In this paper, we construct a bundle on noncommutative toroidal orbifold $\mathbb{T}^4_\theta/\mathbb{Z}_2$. We start with the construction of a bundle on noncommutative $\mathbb{T}^4$ *a la* Rieffel [11] and find a CDS type solution of M(atrix) theory compactified on the noncommutative 4-torus. There, we also show explicitly that the dual tori are actually related to each other through SO(4,4|$\mathbb{Z}$) transformations. Based on our explicit CDS type solution on noncommutative $\mathbb{T}^4$, we find its $\mathbb{Z}_2$ orbifold solutions, **Rep. I** and **Rep. II**, by looking into the systems of D0-branes on the covering space projected onto their invariant parts under the discrete symmetry group. From the solutions obtained, we study the moduli space of equivariant constant curvature

connections. The Higgs branch moduli space has been identified with the ordinary toroidal orbifold in the **Rep. I** case where we consider two copies of a bundle over $\mathbb{T}_\theta$ which are invariant under the $\mathbb{Z}_2$ action on $\mathbb{T}_\theta$. In the **Rep. II** case, the moduli space has no Higgs branch. In conclusion, in the noncommutative $\mathbb{T}^4/\mathbb{Z}_2$ case the moduli space has the same form as its commutative counterpart.

In [16], the moduli space of D0-branes on commutative $\mathbb{T}^4$ with torons of $U(N)$ Yang-Mills theory was given as $(\mathbb{T}^4)^{p_1}/S_{p_1} \times (\mathbb{T}^4)^{p_2}/S_{p_2}$ where $U(N)$ gauge group broken down into $U(k_1) \times U(k_2)$ satisfying $k_1 + k_2 = N$, and $p_i = gcd(k_i, m_i)$, $i = 1, 2$ with fluxes m_i of $U(k_i)$. Its extension to the noncommutative case has been recently studied in [15] using the 't Hooft's $SU(N)$ solution of nontrivial twists [22], and the resulting moduli space of connections turned out to be of the same form, $(\mathbb{T}^4)^{p_1}/S_{p_1} \times (\mathbb{T}^4)^{p_2}/S_{p_2}$. We expect that the same holds for the noncommutative toroidal $\mathbb{Z}_2$ orbifold case.

Note added: After completion of our paper, a related paper [23] has appeared, which has some overlap with our paper. Their methodology to get the relevant moduli spaces is to use the theory of representation of Heisenberg algebra defined by the commutation relations of a fixed connection. On the other hand, our approach is the usual one in that we construct a module on $\mathbb{T}^4_\theta$ with explicit computation, and then consider the $\mathbb{Z}_2$ orbifold condition on this module finding the moduli space in the specific cases.

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